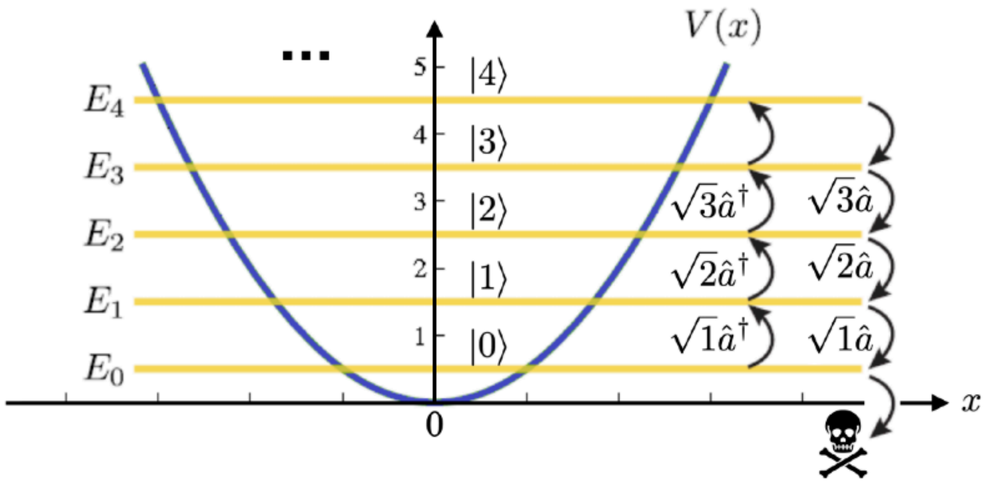
Position eigenstates and wavefunctions

Energy eigenstates

$$\hat{H}|n\rangle = \hbar\omega(n+1/2)|n\rangle$$



Position eigenstates

$$\hat{x}|x = x'\rangle = x'|x = x'\rangle$$

$$\exp\left(-\frac{i\hat{p}\delta x}{\hbar}\right)|x = x'\rangle = |x = x' + \delta x\rangle$$

* Follows from properties of displacement operators

**Act on an arbitrary coherent state to prove

"Position wavefunction of Fock state n"

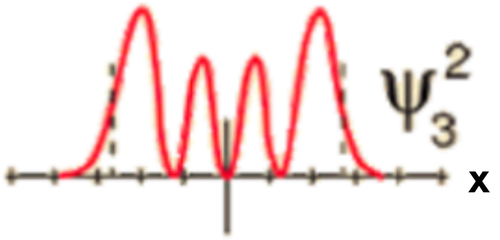
$$|n\rangle = \sum_{x'} \Psi_n(x')|x = x'\rangle$$ energy eigenstate in the position basis

$$|x = x'\rangle = \sum_{n} \Psi_n^*(x')|n\rangle$$ position eigenstate in the energy basis

$$\Psi_n(x') = \langle x = x'|n\rangle \qquad \Psi_n^*(x') = \langle n|x = x'\rangle$$

$$\mathrm{Prob}(x' < x \leq x' + dx') = |\Psi_n(x')|^2 dx' \implies \int_{x=-\infty}^{x=+\infty} |\Psi_n(x)|^2 dx = 1$$

Probability —>
Probability density

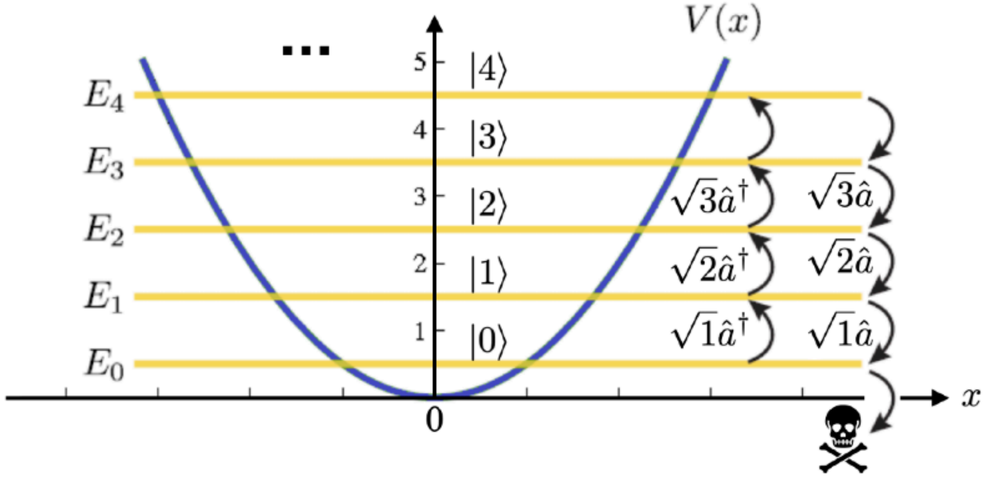


$$\sum_x \Psi_n(x)\Psi_m(x) \to \int_{x=-\infty}^{x=+\infty} \Psi_n(x)\Psi_m(x)dx = \delta_{n,m}$$ **orthogonality**

Wavefunctions of energy eigenstates (Fock states)

Energy eigenstates

$$\hat{H}|n\rangle = \hbar\omega(n+1/2)|n\rangle$$



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position operator

$$\hat{x}|n\rangle = x_0\sqrt{n}|n-1\rangle + x_0\sqrt{n+1}|n+1\rangle = x_0\textstyle\sum_{x'}\Big(\sqrt{n}\Psi_{n-1}(x') + \sqrt{n+1}\Psi_{n+1}(x')\Big)|x = x'\rangle$$

$$\hat{x}|n\rangle = \textstyle\sum_{x'}\Psi_n(x')\hat{x}|x = x'\rangle = \sum_{x'} x'\Psi_n(x')|x = x'\rangle$$

$$\Psi_{n+1}(x) = \frac{1}{\sqrt{n+1}}\Big(\frac{x}{x_0}\Psi_n(x) - \sqrt{n}\Psi_{n-1}(x)\Big)$$

Recursion

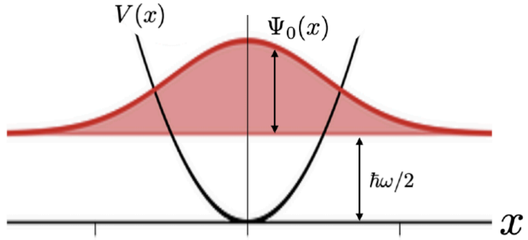
$$\Psi_{-1} = 0$$

momentum operator

$$\hat{p}|n\rangle = \hat{p}\sum_x \Psi_n(x)|x\rangle = \sum_x(-i\hbar)\frac{\partial\Psi_n(x)}{\partial x}|x\rangle$$

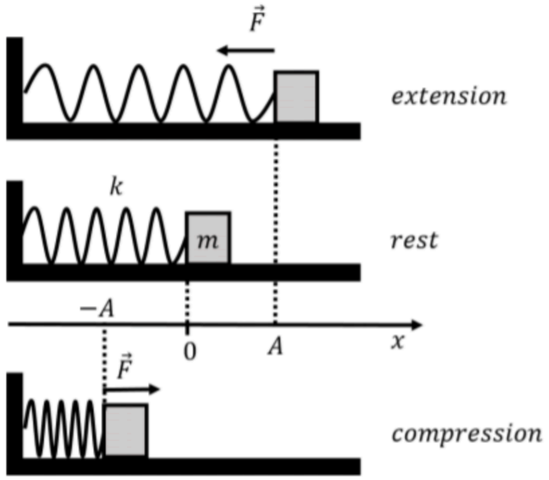
$$\hat{a}|0\rangle = \Big[\frac{\hat{x}}{2\hat{x}_0} + \frac{i\hat{p}}{2p_0}\Big]|0\rangle = 0$$

$$x\Psi_0(x) + 2x_0^2\frac{\partial\Psi_0(x)}{\partial x} = 0$$



$$\Psi_0(x) = \frac{1}{(2\pi x_0^2)^{1/4}}\exp\Big[-(x/2x_0)^2\Big]$$

Where is the oscillator?



$$\Psi_3(x) = \frac{1}{\sqrt{6}}(x/x_0)\left[(x/x_0)^2 - 3\right]\Psi_0(x)$$

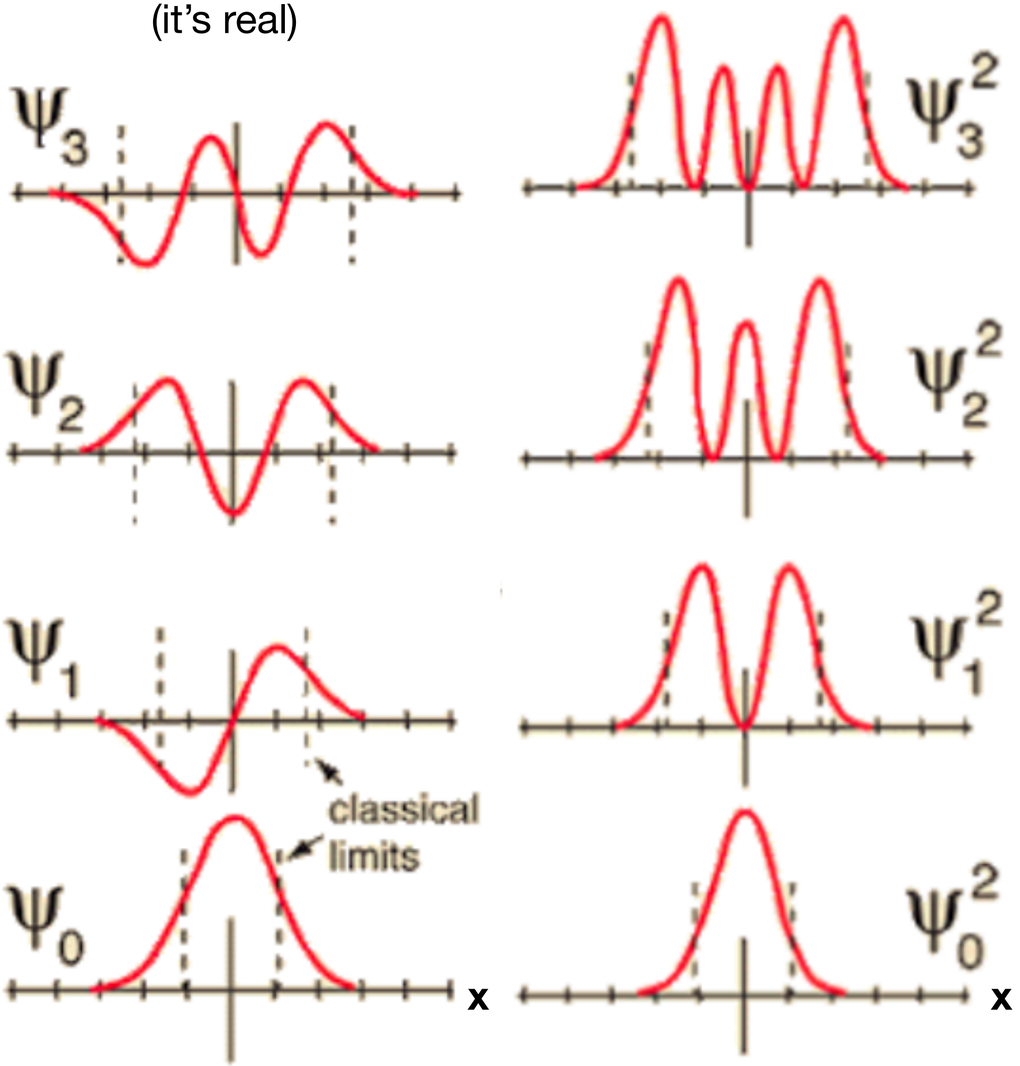
$$\Psi_2(x) = \frac{1}{\sqrt{2}}\left[(x/x_0)^2 - 1\right]\Psi_0(x)$$

$$\Psi_1(x) = (x/x_0)\Psi_0(x)$$

$$\Psi_0(x) = \frac{1}{(2\pi x_0^2)^{1/4}}\exp\left[-(x/2x_0)^2\right]$$

Wavefunction
(it's real)

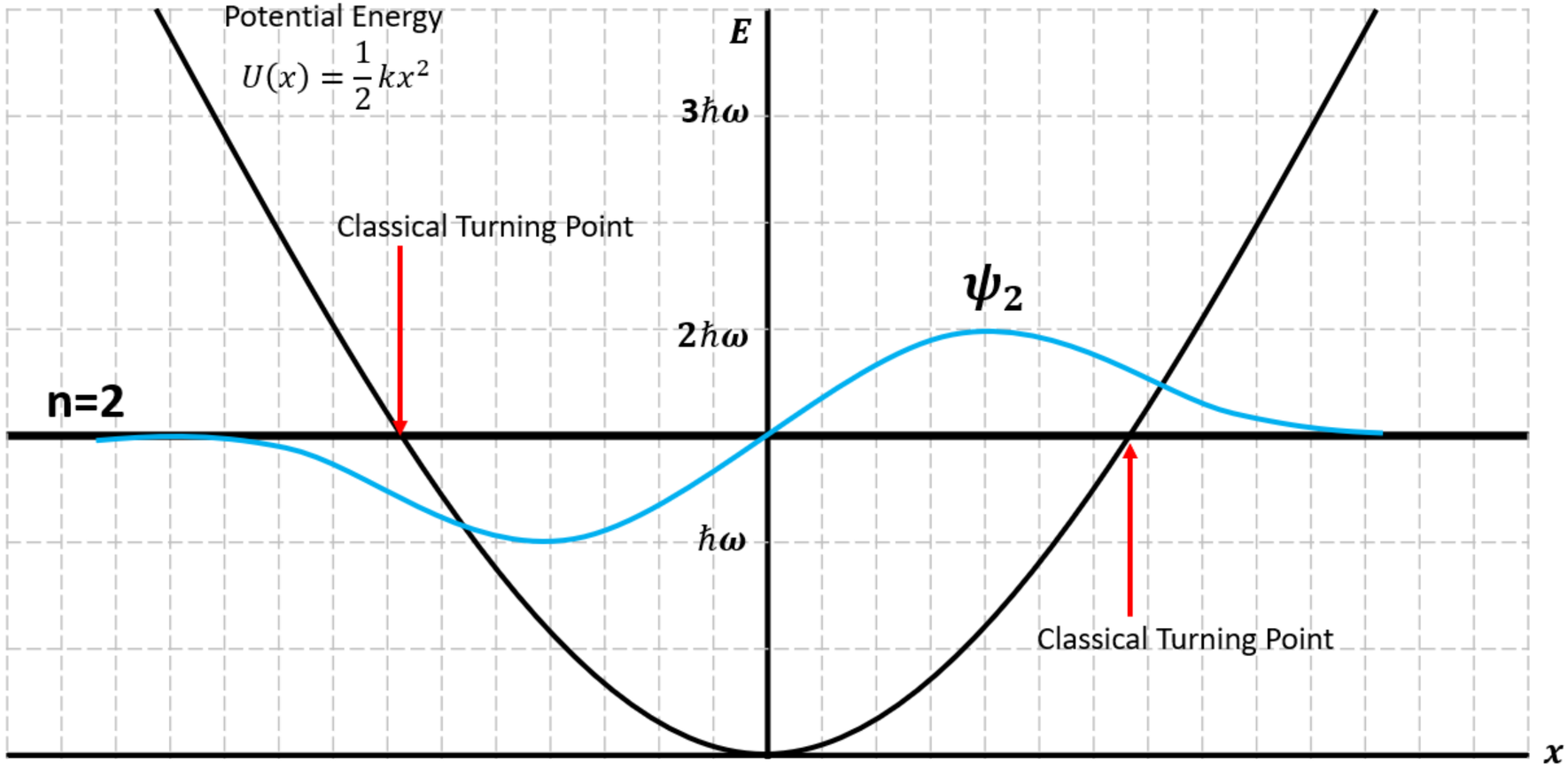
Probability density



Ground state corresponds to "motion" —> quantum fluctuations
Excited states oscillate more the higher the energy

x_0 is one unit

Where is the oscillator? It can be where it cannot be classically!

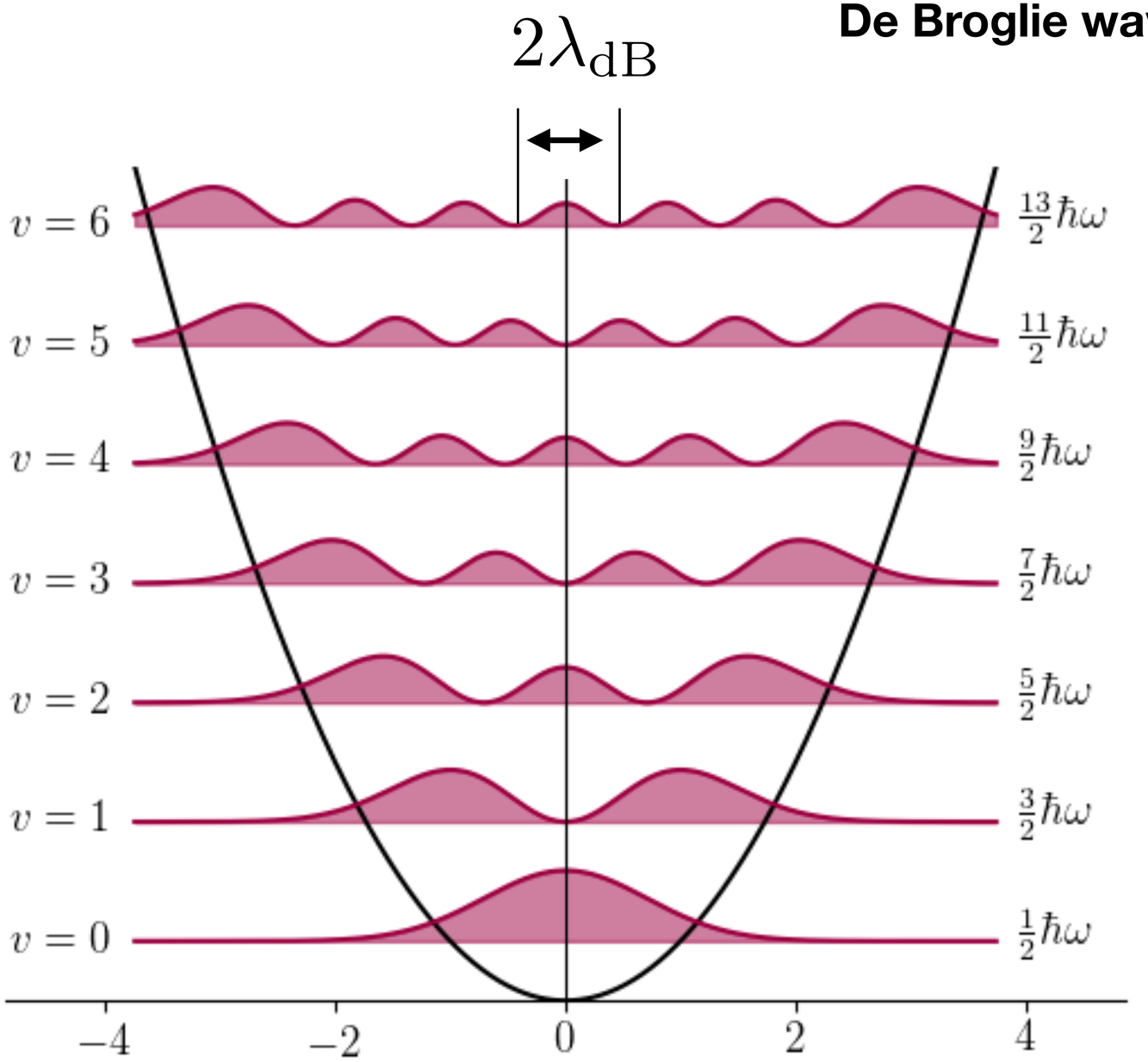


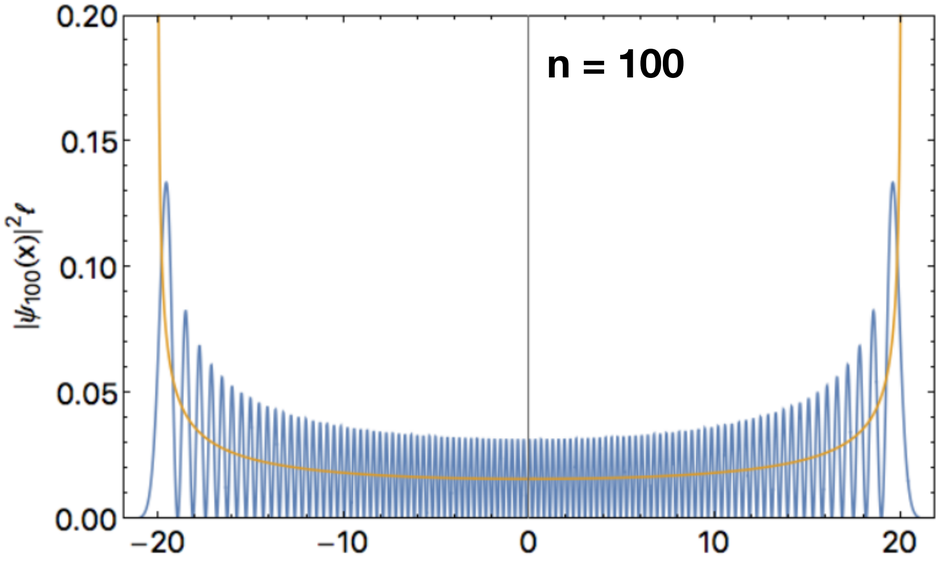
Classical turning point

$$p^2/2m = E - m\omega^2 x_c^2/2 = 0$$

$$\text{Prob}(|x| > x_c) \approx 10\%$$

De Broglie wavelength





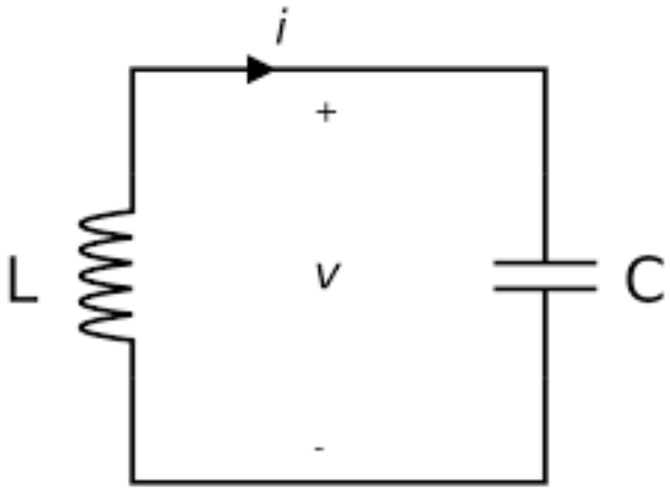
$$x \approx 0 \implies U(x) \ll E_n \implies |p| = \sqrt{2mE_n}$$

!!particle = wave!! $\lambda_{\mathrm{dB}} = h/p$

The higher the energy the harder to see the wave!

From oscillator to a qubit

Electrical circuits are oscillators too!



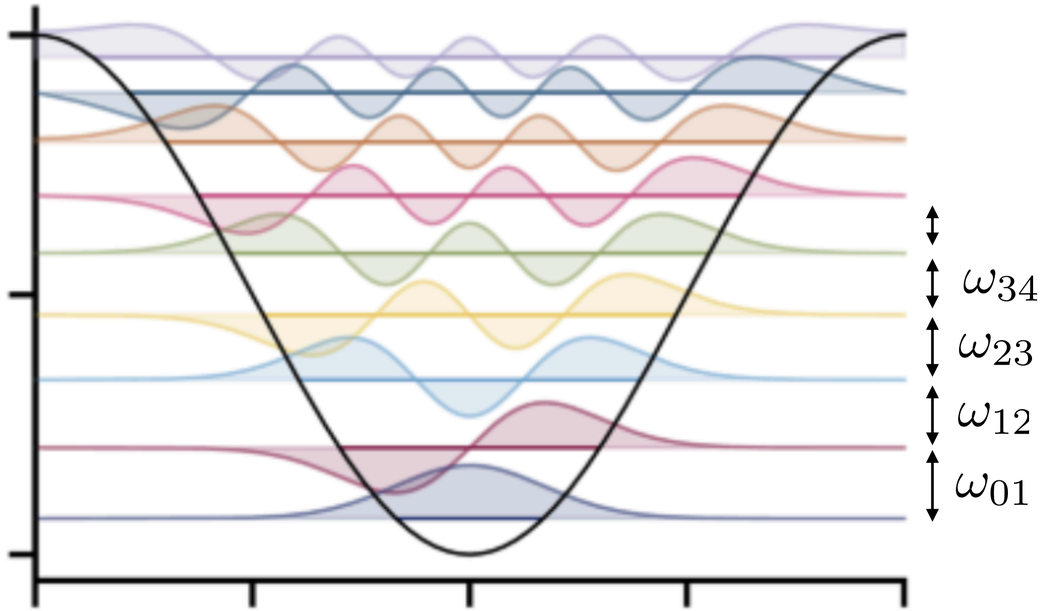
$$p \rightarrow \text{charge}$$

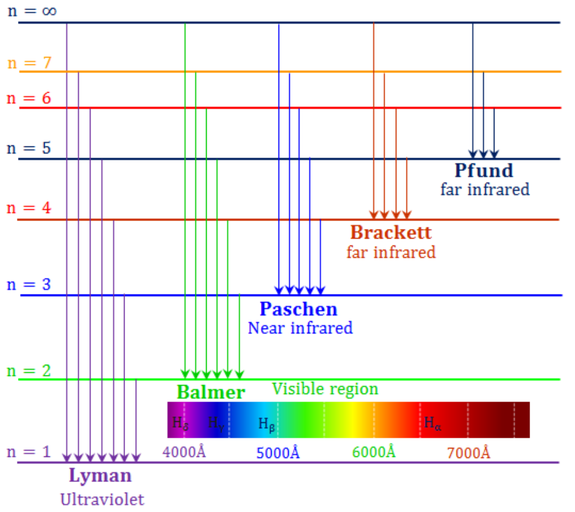
$$x \rightarrow \text{flux}$$

$$m \rightarrow \text{capacitance}$$

$$\omega \rightarrow 1/\sqrt{LC}$$

$$[\text{flux}, \text{charge}] = i\hbar$$





Engineer your own "atom" at SQIL!

non-linear inductance

$$H = p^2/2m + U(x) \approx \underbrace{p^2/2m + m\omega^2 x^2/2}_{\text{diagonal matrix}} + \underbrace{Kx^4}_{\text{non-diagonal matrix}}$$

$$+\hat{H}_{drive} = g\hat{x}\cos(\omega_{01}t) \rightarrow \text{anharmonic spectrum} \rightarrow \text{qubit}$$

Wave functions and time evolution of an energy superposition state

$$|\Phi\rangle = \sum_{x'} \Phi(x')|x = x'\rangle \qquad (67)$$

Some initial state with a known wavefunction

How to work out this evolution? The same ket can be decomposed over the Fock states:

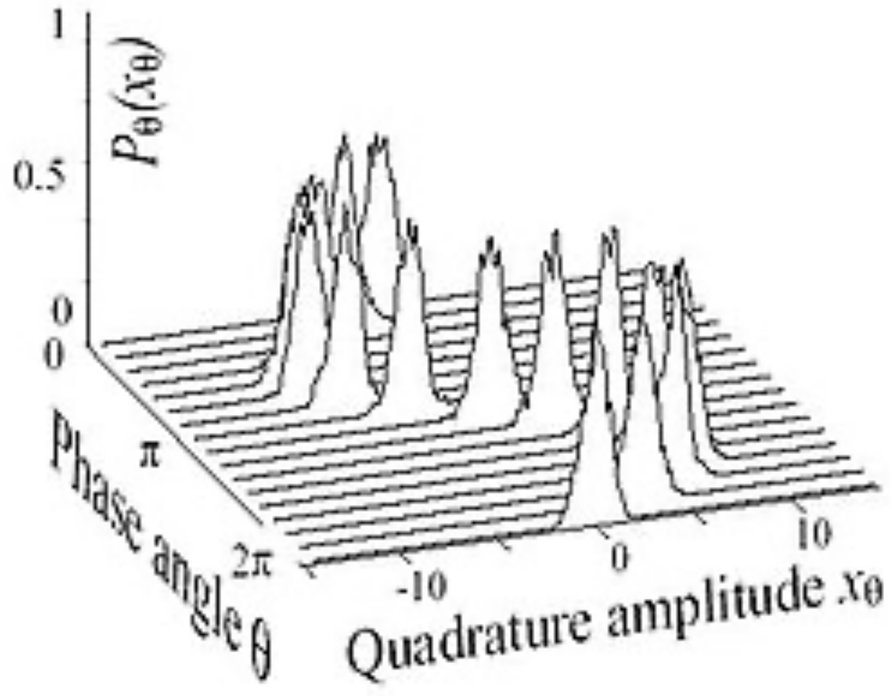
$$|\Phi\rangle = \sum_{n} \phi_n |n\rangle \qquad (68)$$

Therefore,

$$\Phi(x) = \langle x|\Phi\rangle = \sum_{n} \phi_n \Psi_n(x) \qquad (69)$$

$$\Phi(x,t) = \sum_{n} \phi_n \exp(-in\omega t)\Psi_n(x)$$

Example: coherent state



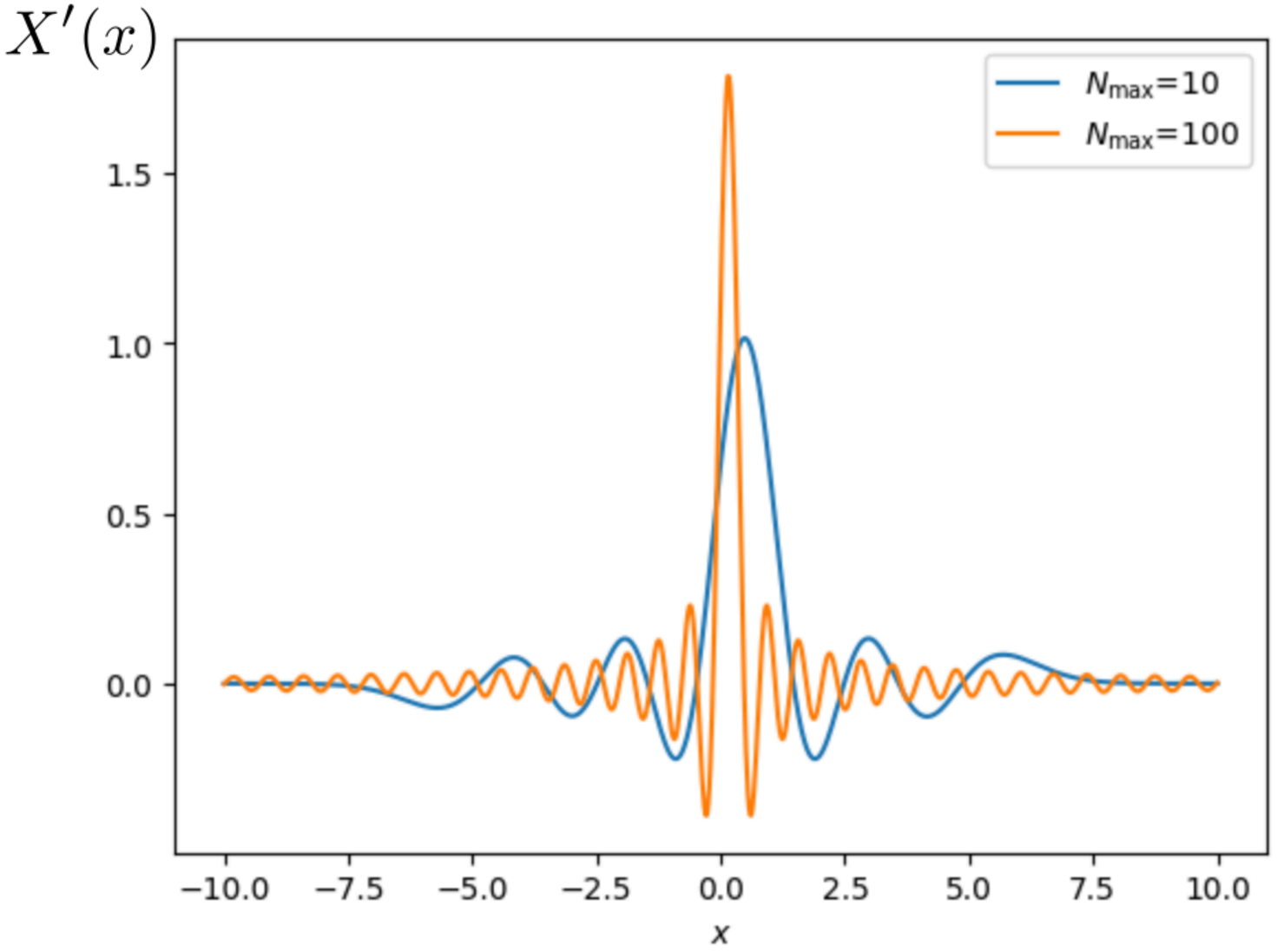
$$|\Phi\rangle = \sum_n \phi_n |n\rangle \qquad \phi_n = \alpha^2/\sqrt{n!}$$

$$\Phi(x,t) = \sum_n \frac{\Big(\alpha \exp(-i\omega t)\Big)^n}{\sqrt{n!}} \Psi_n(x)$$

Watch movie here

https://en.wikipedia.org/wiki/File:Coherent_state_gif.gif

What is wave-function of a position eigenstate?



Find numerical eigenvectors of

$$\hat{x} = x_0(\hat{a} + \hat{a}^\dagger)$$

$$\hat{a}|\Psi\rangle = \begin{pmatrix} 0 & \sqrt{1} & 0 & 0 & \dots \\ 0 & 0 & \sqrt{2} & 0 & \dots \\ 0 & 0 & 0 & \sqrt{3} & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & & & & \end{pmatrix} \begin{pmatrix} \psi_0 \\ \psi_1 \\ \psi_2 \\ \psi_3 \\ \dots \end{pmatrix}$$

$$\hat{x}|x = x'\rangle = x'|x = x'\rangle$$

$$|x = x'\rangle = \sum_{n=0}^{N_{max}} \phi_n(x')|n\rangle$$

eigenvector for eigenvalue x'

$$X'(x) = \sum_{n=0}^{N_{max}} \phi_n(x')\Psi_n(x)$$

Approximate wavefunction for position eigenstate with eigenvalue x'
The higher Nmax, the more accurate the wavefunction is!